Counting Magic Squares

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Let us consider the following distribution problem.
Assume that we have to distribute 90 building blocks among 3 students. Let 30 blocks are white, 30 blocks are red, and 30 blocks are green. our aim to distribute the given blocks among the 3 students equally. So, the question is in how many ways can this be done if blocks of the same color are identical?

Defn: A magic square is a square matrix with non-negative entries in w/c all rows and all columns have the same sum.

- **Example:** $\begin{array}{|c|c|}\hline 4 & 3 \\ \hline 3 & 4 \\ \hline\end{array}$ is a magic square

Let us call the rows and colums with the same word lines.
Let $H_n(r)$ be the # of magic squares of size $n \times n$ having the line sum r.
Let us consider small size of n.

$n=1$: $\boxed{r}$, $H_1(r) = 1$

$n=2$: $\begin{array}{|c|c|}\hline x & r-x \\ \hline r-x & x \\ \hline\end{array}$, $0 \le x \le r$. Then x has $r+1$ possibilities.

$\therefore H_2(r) = r+1$.

$n=3$: We have the following theorem.

Theorem (P. MacMhonian, 1916)
For all non-negative integers, $H_3(r) = \binom{r+4}{4} + \binom{r+3}{4} + \binom{r+2}{4}$ - - - - ①

proof: To prove this theorem, we have to follow other way other than that of $n=2$.

$$\begin{array}{|c|c|c|}\hline a & d & ① \; r-a-d \\ \hline ⑤ \; r+c-(a+b+d) & b & ③ \; a+d-c \\ \hline ④ \; b+d-c & ② \; r-b-d & c \\ \hline\end{array}$$

Box ①: $r-a-d \ge 0 \Rightarrow a+d \le r$ - - - - ②
Box ②: $r-b-d \ge 0 \Rightarrow b+d \le r$ - - - - ③
Box ③: $a+d-c \ge 0 \Rightarrow c \le a+d$ - - - - ④
Box ④: $b+d-c \ge 0 \Rightarrow c \le b+d$ - - - - ⑤
Box ⑤: $r+c-(a+b+d) \ge 0 \Rightarrow a+b+d-c \le r$ - - - ⑥

[The number of integral solutions to such system of linear equations is the sum of the three binomial coefficients].
Let us split the set of all magic square of size 3×3 of line sum r into 3 subsets.

1) The subset of all magic squares of size 3×3 in which $a\le b$ & $a\le c$.

2) The subset of all magic squares in which $b<a$ & $a\le c$.

3) The subset of all magic squares in which $c<a$ & $c<b$.

We can enumerate the magic squares in each subset by combining all conditions ② - ⑥ into one chain of inequalities in order to improve our chances of counting their sums.

Now to prove the theorem, let us start by case - 1.

(1) $a\le b$ & $a\le c$ with 3 & 4

$$\Rightarrow a \overset{4}{\le} 2a+d-c \overset{a\le b}{\le} a+b+d-c \overset{a\le c}{\le} b+d \overset{3}{\le} r \quad \text{- - - - - ⑦}$$

Our counting problem is reduced to the number of 4-tuples of non-negative integers (a,b,c,d) for which ⑦ holds.

There is a bijection b/n (a,b,c,d) and $(a, 2a+d-c, a+b+d-c, b+d)$

If $S=\{0,1,\dots,r-1,r\}$, then we have to choose 4 elements out of $r+1$ elements of S with repetition.

$$\Rightarrow \binom{(r+1)+(4-1)}{4} = \binom{r+4}{4} \text{ ways are there}$$

(2) and (3) are exercises!

Case-2: $b\le 2b+d-c \le a+b+d-c-1 \le a+d-1 \le r-1$

Case-3: $c\le b-1 \le b+d-1 \le a+b+d-c-2 \le r-2$. ∎

Properties of $H_n(r)$

- $H_n(r)$ is a polynomial f/n of r for any fixed +ve integer n.

Theorem: For any fixed +ve integer n, the degree of $H_n(r)$ is $(n-1)^2$.

proof: Recall that if two polynomials p and q both having the same degree m, satisfy $p(i)=q(i)$ in $m+1$ d/t points, then p & q are identical.

First we show that $\deg(H_n(r)) \le (n-1)^2$. To show this it suffices to show that there is a polynomial of degree $(n-1)^2$ whose values are always larger than the corresponding values of $H_n(r)$.

An $n\times n$ magic squares is completely determined by the $n-1\times n-1$ elements of its top left corner. Each of these elts can range from 0 to r, so there are $r+1$ choices for each of these elts.

Thus $H_n(r) \le (r+1)^{(n-1)^2}$

$$\therefore \deg(H_n(r)) \le (n-1)^2 \quad \text{- - - - - ①}$$

Next, we show that $\deg(H_n(r)) \geq (n-1)^2$.

Here we need to show that there exists a polynomial of degree $(n-1)^2$ whose values are always smaller than the corresponding values of $H_n(r)$ and whose leading coefficient is positive.

Consider

$$\begin{array}{|c|c|c|c|c|}
\hline
* & * & * & \cdots & r-R_1 \\ \hline
* & * & * & \cdots & r-R_2 \\ \hline
* & * & * & \cdots & r-R_3 \\ \hline
\vdots & \vdots & \vdots & \ddots & \vdots \\ \hline
r-c_1 & r-c_2 & r-c_3 & \cdots & a \\ \hline
\end{array}$$

$a = s-(n-2)r$

To find such a polynomial, let us try to find many diff ways to fill the top left corner $(n-1 \times n-1)$ subsquare matrix so that each case leads us to a valid magic square.

Take the above magic square, where

R_i = the sum of $(n-1)\times(n-1)$ i^{th} row

C_i = the sum of $(n-1)\times(n-1)$ i^{th} column

s = the sum of total elts in $(n-1)\times(n-1)$ sub square

$s+2r = nr+a \Rightarrow a = s+2r-2n = s-(n-2)r.$

As the above figure shows i) $\forall i \in [1, n-1]$, $R_i, C_i \leq r$

ii) $(n-2)r \leq s$.

(i) $\Rightarrow$ each of the $(n-1)^2$ entries always less than or equal to $r/_{n-1}$.

(ii) $\Rightarrow$ the minimum value of each entry of the $(n-1)\times(n-1)$ sub-square is $(n-2)r/(n-1)^2$.

The total sum s of $(n-1)^2$, if each entry is a, then

$$s = a(n-1)^2 \leq (n-1)r$$

$$\Rightarrow a \leq (n-1)r/(n-1)^2 = r/_{n-1}$$

Again $nr \leq 2r+s \iff nr \leq 2r + a(n-1)^2$

$$\iff nr-2r \leq a(n-1)^2$$

$$\iff (n-2)r \leq a(n-1)^2$$

$$\iff \frac{(n-2)r}{(n-1)^2} \leq a$$

$\therefore$ We get a magic square each time we choose each entry t of the top left corner $(n-1)\times(n-1)$ such that $\frac{(n-2)r}{(n-1)^2} \leq t \leq \frac{r}{n-1}$

The # of such choices for each entry is at least

$$\frac{r}{n-1} - \frac{(n-2)r}{(n-1)^2} = \frac{r}{(n-1)^2} - 1$$

$$\therefore H_n(r) \geq \left(\frac{r}{(n-1)^2} - 1\right)^{(n-1)^2}$$

$$\Rightarrow \deg(H_n(r)) \geq (n-1)^2 \quad \text{-----} \quad (2)$$

Therefore, from (1) and (2),

$$\deg(H_n(r)) = (n-1)^2$$

Note: The magic squares of line sum 1 are those matrices containing exactly 1 in each row & column and zero elsewhere. that is, they are permutation matrices.

Proposition: There is a bijection b/n $n \times n$ magic square of line sum 1 and permutations of length n.

In particular, $H_n(1) = n!$

Proof: Exercise!

- Decomposition of magic squares

Any magic square can be decomposed into sum of permutation matrices

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Lemma: (Birkoff Von Newman Theorem)

Any magic square of line sum r can be decomposed into sum of r permutation matrices. (Not necessarily unique)

$$\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix} = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} + \begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} + \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{matrix} = \begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix} + \begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix} + \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{matrix}$$

For $n \geq 3$, there are d/t sets of $n \times n$ permutation matrices that sum to the same magic square.

If the decomposition were unique we could argue that "since the number of permutation matrices on size n is $n!$" we have to choose r of them, not necessarily d/t ones and add them.

This can be done in $\binom{n!+r-1}{r} = \binom{n!+r-1}{n!-1}$ ways and that is a polynomial of r having degree $n!-1$.

However, since the decomposition is not unique, this argument leads us to over count. To correct the mistake caused by over counting we will have to see that the over count itself is a polynomial and therefore the actual fln $H_n(r)$ is a difference of the two polynomials.

i.e, it is the dize of the total decomposition and the surplus, (the sum of the # of decompositions each magic square has in excess of one.)

<u>Theorem</u>: For any integer r (non-negative)

$$H_3(r) = \binom{r+5}{5} - \binom{r+2}{5}$$

<u>Proof</u>: Let H be a 3×3 magic square.

The # of 3×3 permutation matrices $= 3! = 6$.

The # of decompositions of H = the # of ways to choose r of these 6 permutations with repetition allowed.

$$= \binom{3!+r-1}{r} = \binom{r+5}{r} = \binom{r+5}{5}$$

It is routine to check that any ~~of~~ 5 of these 6 permutation matrices $P_1, P_2, \ldots, P_6$ are linearly independent.

Therefore, the magic square H can have more than one decomposition only if one can subtract each of the 6 P_i from and still get a magic square.

In other words, if H consists of the entries only, then H has more than one decomposition.

Indeed, we can subtract the matrix $J_3 = \begin{matrix}1&1&1\\1&1&1\\1&1&1\end{matrix}$ to get

$H = J_3 + H'$

We have seen that J_3 has 2 decompositions

$$J_3 = \{P_1 + P_2 + P_3\} = \{P_4 + P_5 + P_6\}$$

So, H can have at least 2 decompositions. It remains to count the surplus decompositions.

For instance, take a decomposition of A in w/c each element of P_4, P_5 and P_6 occurs atleast once.
Using the identity $P_1 + P_2 + P_3 = P_4 + P_5 + P_6$ we can replace a subset $\{P_4, P_5, P_6\}$ in this decomposition by a subset $\{P_1, P_2, P_3\}$ and get a new decomposition.
We can iterate this procedure as long as each of P_4, P_5, P_6 occurs in the decomposition atleast once after w/c we have to stop.
Therefore, we will say that a decomposition is surplus iff the coefficients of each P_4, P_5, P_6 are all positive in the decomposition.
i.e., ⊛ holds if we choose P_1, P_2, P_3.
To choose r permutation matrices out of these 6 whose 3 of them are determined to be either $\{P_1, P_2, P_3\}$ or $\{P_4, P_5, P_6\}$.
Then the # of such decomposition is

$$\binom{3! + r - 3 - 1}{r-3} = \binom{3! + r - 3 - 1}{r-3}$$

$$= \binom{r+2}{r-3} = \binom{r+2}{5}$$

$$\therefore H_3(r) = \text{Total decomposition} - \text{surplus decomposition}$$

$$= \binom{r+5}{5} - \binom{r+2}{5}$$

Hence, $H_3(r) = \binom{r+4}{4} + \binom{r+3}{4} + \binom{r+2}{4} = \binom{r+5}{5} - \binom{r+2}{5}$ ⊛