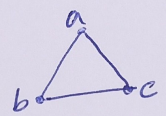
Futa Beyene

Counting k-colored graphs

Chromatic polynomial

- Coloring of a graph is an assignment of colors to the vertices a graph such that any adjacent vertices receive different colors.
- k-Coloring is a coloring using at most k colors.
- The Smallest number of colors needed to color the graph G is called the chromatic number, $\chi(G)$.
- Example: Let $G = K_3$. Then $\chi(G) = 3$.



If the three colors are c_1, c_2, & c_3, then

a	c_1	c_1	c_2	c_2	c_3	c_3
b	c_2	c_3	c_1	c_3	c_1	c_2
c	c_3	c_2	c_3	c_1	c_2	c_1

Thus, we can color K_3 in 6 dIt ways.

- We can ask: what would be the total number of colorings if we have k colors, where $k \geq 3$:

Then the answer is $k(k-1)(k-2) = k(k^2-3k+2) = k^3-3k^2+2k$

Defn: The chromatic polynomial of graph G is the number of different k-colorings, $P_G(k)$.

- **Example**: 1. The chromatic polynomial of $G = K_n$ is given by

$$P_G(k) = k(k-1)(k-2)\ldots(k-(n-1))$$

2. The chromatic polynomial of $G = N_n$ - null graph is

$$P_G(k) = k^n.$$

The three operations

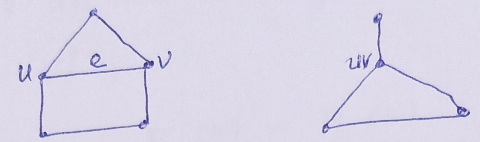
i. Addition of an edge: $G+uv$ — a graph with one new edge

ii. Deletion of edge uv: $G-uv$ — a graph without the old edge uv.

iii. Contraction of two vertices u and v.

- Without the old vertices u & v and the edges that are connected between them.
- With a new vertex uv that is connected to the neighbors of u & v.

- Example:



G $\qquad$ $G/(u,v)$

Recursive formulas for $P_G(K)$:

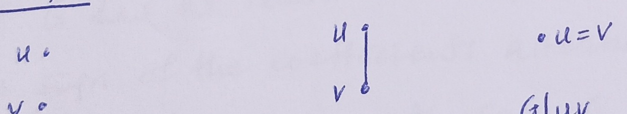
Theorem-1: For any non-adjacent vertices u and v, we have

$$P_G(K) = P_{G+uv}(K) + P_{G/uv}(K)$$

Proof: Let u and v be two vertices of G. Then either they receive d/t colors or the same color. If they receive d/t colors, then we can add an edge b/n u & v and this contribut $P_{G+uv}(K)$. If they receive the same color, then they are non-adjacent and we can contract so that the resulting graph will have chromatic polynomial $P_{G/uv}(K)$.

Thus, $P_G(K) = P_{G+uv}(K) + P_{G/uv}(K)$ ∎

- Example: Let $G = N_2$.



G $\qquad$ $G+uv$ $\qquad$ G/uv

$P_G(K) = K^2$ $\qquad$ $P_{G+uv}(K) = K(K-1) = K^2 - K$ $\qquad$ $P_{G/uv}(K) = K$

$\Rightarrow P_G(K) = K^2 = K^2 - K + K$

Theorem-2: Fundamental Reduction Formula

For any edge uv,

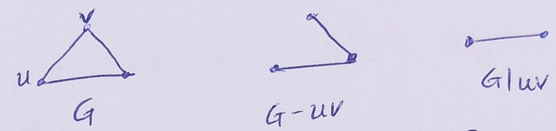
$$P_G(K) = P_{G-uv}(K) - P_{G/uv}(K)$$

Proof: $P_{G-uv}(K)$ covers all the colorings in wic the color of u is the same as the color of v and all the colorings in wic the color of u is different from the color of v.

$P_{G/uv}(K)$ covers all the colorings in wic the color of u is the same as the color of v.

$$\therefore P_G(K) = P_{G-uv}(K) - P_{G/uv}(K) \quad \#$$

- **Example:** let $G = K_3$.



$$P_G(K) = K(K-1)(K-2) = K^3 - 3K^2 + 2K$$

$$P_{G-uv}(K) = K(K-1)^2 = K^3 - 2K^2 + K$$

$$P_{G/uv}(K) = K(K-1) = K^2 - K$$

$$\Rightarrow P_G(K) = K^3 - 3K^2 + 2K = K^3 - 2K^2 + K - K^2 + K$$

Properties of $P_G(K)$

- $P_G(K) = a_n K^n + a_{n-1} K^{n-1} + \dots + a_0$

i) $a_0 = 0$, ii) $a_n = 1$, iii) $|a_{n-1}| = \|G\|$

iv) If G has at least one edge, then $a_n + a_{n-1} + \dots + a_0 = 0$.

v) The sign of the coefficients alternate between +ve & -ve.

vi) Let G has h components, say, $G_1, G_2, \dots, G_h$, then

$$P_G(K) = P_{G_1}(K)\, P_{G_2}(K) \cdots P_{G_h}(K)$$

Proof: If G has $m = 1$ edge, then $P_G(1) = 0$ and $a_n + a_{n-1} + \dots + a_0 = 0$

Suppose that G has m edges. Then we use induction on m.

1) $m = 0 \Rightarrow P_G(K) = K^n = K^n - 0K^{n-1} + \dots$ wic is true.

2) Assume that it is true for $m-1$ or less.

$$P_G(K) = P_{G-uv}(K) - P_{G/uv}(K)$$

Let $uv \in E(G)$. Consider $P_{G-uv}(K)$ and $P_{G/uv}(K)$

Since the edges of $G-uv$ is reduced by 1

$$P_{G-uv}(k) = k^n - a_{n-1}k^{n-1} + \cdots + a_1 k \quad \text{--- by (2)}$$

$$P_{G/uv}(k) = k^{n-1} - b_{n-2}k^{n-2} + \cdots$$

$$\Rightarrow P_G(k) = P_{G-uv}(k) - P_{G/uv}(k)$$

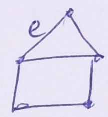
$$= k^n - (a_{n-1}+1)k^{n-1} + (a_{n-2}+b_{n-2})k^{n-2} + \cdots$$

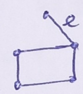
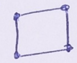
<u>Reduction Algorithm</u>

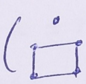
Let $e \in G$, then set $G'_e := G-e$ and $G''_e := G/e$. Then

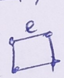
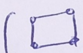
$$P_G(k) = P_{G'_e}(k) - P_{G''_e}(k)$$

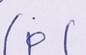
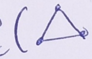
By Theorem-1, $P_{G'_e}(k) = P_G(k) + P_{G''_e}(k)$

- <u>Example</u>: Find $P_G(k)$, where $G =$ 

<u>Sol^n</u>: $P_G(k) = P_k($  $) - P_k($  $)$

$= P_k($  $) - 2P_k($  $)$

$= k\,P_k($  $) - 2P_k($  $)$

$= (k-2)\big(P_k($  $) - P_k($  $)\big)$

$$= (k-2)\left(k(k-1)^3 - k(k-1)(k-2)\right)$$

$$= (k-1)(k-2)\left[k(k-1)^2 - k(k-2)\right]$$

- Continue until you get any of the following two known chromatic polynomials.

<u>Remark</u>: ① $P_k(K_n) = k(k-1)\cdots(k-(n-1))$

② $P_k(T_n) = k(k-1)^{n-1}$, T_n - a tree on n-vertices.