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Counting polyominoes

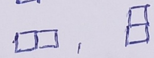
Defn: i. A cell is the interior and boundary of a unit square in the xy-plane, if the vertices of the square are at lattice points (points whose coordinates are both integers).

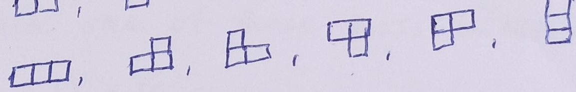
ii. Let P be a collection of cells.
We associate with P a graph, whose vertices are joined by an edge in the graph if the two cells they correspond intersect in a line segment (rather than in a vertex, or not at all)
- P is said to be connected if the graph associated with P is a connected graph.
- P is in standard position if all of its cells lie in the 1st quadrant, and at least one of them intersect the y-axis and at least one of them intersect the x-axis.

iii. A polyomino is a connected collection of cells that is in standard position.

Let n = # of cells and $f(n)$ = # of polyominoes in the collection P of n cells. (# of n-celled polyominoes).

Eg. $n=1$, $f(1)=1$: 

$n=2$, $f(2)=2$: 

$n=3$, $f(3)=6$: 

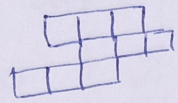
Some properties of polyomino

- Various special kinds of polyominoes, however, have been counted. w.r.t. various properties of polymino.

For instance, among the properties that a polyomino has one might mention its area, or numbers of celles, and its perimeter.

iv. Horizontally convex (HC) - polyomino is a polyomino in which every row is a single contiguous block of cells.

eg.



v. Convex polyomino is a polyomino in w/c it is both horizontally and vertically convex.

- There are interesting problems involved in counting HC- polyominoes either by area or by perimeter.

Que: How many HC - polyominoes of area n are there?

Let $f(n, k, t)$ = # of HC - polyominoes of n cells, having k rows, of which t are in the top row.

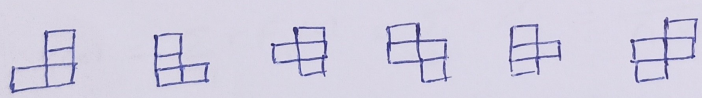
Step-1: Strip off the top row of one these polyominoes.
So, what will remain will have $n-t$ cells, arranged in $k-1$ rows, with some number $r \geq 1$ in the top row.
Hence, after removing the top row, there are

$$f(n-t, k-1, r)$$

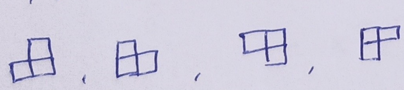
possibilities for some r.

Step-2: Examine each one of those possibilities.

Eg. Consider $f(4, 3, 1) = 6$



By Step-1, removing the top row of 1 cell, we get



$\Rightarrow 2f(3,2,1)$ and $2f(3,2,2)$

- Adjoin a top row 1 cell and slide it right and left through all legal positions at the top of the second row, we have

$$1+1+2+2 = 6$$

$\Rightarrow$ each contribute $(1+1-1)$ of the original, $r = 1, 2$

Hence, in general, each one of the possibilities in step 1 generates $r+t-1$ of the original (n,k,t) HC-polyominoes.

Thus, from step 1 and 2, we have

$$f(n,k,t) = \sum_{r\geq 1} f(n-t,k-1,r)\,(r+t-1)\,,\ k\geq 2,\ f(n,1,t) = \delta_{t,n}\,, \quad ---\ ①$$

Where $\delta_{t,n} = \begin{cases} 1, & \text{if } t=n \\ 0, & \text{otherwise} \end{cases}$

Define the g.f. $F_{k,t}(x) = \sum_{n} f(n,k,t)\,x^n$.

Then $F_{1,t}(x) = \sum f(n,1,t)\,x^n = \sum \delta_{t,n}\,x^n = x^t,\ t\geq 1$

From ①, $f(n,k,t) = \sum_{r\geq 1} f(n-t,k-1,r)\,(r+t-1)$

Applying the snake-oil method, we have

$$\sum_{n} f(n,k,t)\,x^n = \sum_{n}\Big(\sum_{r\geq 1} f(n-t,k-1,r)\,(r+t-1)\Big)\,x^n$$

$$\Leftrightarrow F_{k,t}(x) = \sum_{r\geq 1}(r+t-1)\,x^t\sum_{n} f(n-t,k-1,r)\,x^{n-t}$$

$$= \sum_{r\geq 1}(r+t-1)\,x^t\,F_{k-1,r}(x)\,,\ k\geq 2 \quad ---\ ②$$

Let $U_k(x) = \sum_{r\geq 1} F_{k,r}(x)$ and $V_k(x) = \sum_{r\geq 1} r\,F_{k,r}(x)$.

Then $U_1(x) = \sum_{r\geq 1} F_{1,r}(x) = \sum_{r\geq 1} x^r = \frac{x}{1-x}$ and

$$V_1(x) = \sum_{r\geq 1} r\,F_{1,r}(x) = \sum_{r\geq 1} r\,x^r = \frac{x^2}{(1-x)^2}$$

Further (2) $\Rightarrow F_{k,t}(x) = x^t\Big[\sum_{r\geq 1} r\,F_{k-1,r}(x) + \sum_{r\geq 1}(t-1)\,F_{k-1,r}(x)\Big]$

$$= x^t\big[V_{k-1}(x) + (t-1)\,U_{k-1}(x)\big]$$

$$\Leftrightarrow F_{k,t}(x) = x^t\big(V_{k-1}(x) + (t-1)\,U_{k-1}(x)\big)\,,\ k\geq 2 \quad ---\ ③$$

Taking the sum on t, we have

$$\sum_t F_{k,t}(x) = \sum_t \left(V_{k-1}(x) + (t-1) U_{k-1}(x) \right) x^t$$

$$\Leftrightarrow U_k(x) = V_{k-1}(x) \sum_t x^t + U_{k-1}(x) \sum_t (t-1) x^t$$

$$= \frac{x}{1-x} V_{k-1}(x) + \frac{x^2}{(1-x)^2} U_{k-1}(x) \quad \text{-------- (4)}$$

If we first multiply (3) by t and then sum on t, we have

$$\sum_t F_{k,t}(x)\, t = \sum_t t x^t \left(V_{k-1}(x) + (t-1) U_{k-1}(x) \right)$$

$$\Leftrightarrow V_k(x) = \frac{x}{(1-x)^2} V_{k-1}(x) + \frac{2x^2}{(1-x)^3} U_{k-1}(x) \quad \text{---- (5)}$$

Now we have two simultaneous recurrence to solve for the sequences U_k and V_k.

From (4) solve for $V_{k-1}(x)$ in terms of U_k and U_{k-1}.

$$\Rightarrow V_{k-1}(x) = \frac{1-x}{x} U_k(x) - \frac{x}{1-x} U_{k-1}(x)$$

Substituting in (5), we have

$$V_k(x) = \frac{x}{(1-x)^2} \left(\frac{1-x}{x} U_k(x) - \frac{x}{1-x} U_{k-1}(x) \right) + \frac{2x^2}{(1-x)^3} U_{k-1}(x)$$

$$= \frac{1}{1-x} U_k + \frac{x^2}{(1-x)^3} U_{k-1}(x)$$

Now we consider $U_{k+1}(x) = \sum_{r \geq 1} F_{k+1,r}(x)$

$$\Leftrightarrow U_{k+1}(x) = F_{k+1,1}(x) + F_{k+1,2}(x) + F_{k+1,3}(x) + \cdots$$

$$= x \sum_{r\geq 1} (r+1-1) F_{k,r}(x) + x^2 \sum_{r \geq 1} (r+2-1) F_{k,r}(x) + x^3 \sum_{r\geq 1} (r+3-1) F_{k,r}(x)$$

$$+ \cdots$$

$$= x \sum_{r\geq 1} r F_{k,r}(x) + x^2 \sum_{r\geq 1} (r+1) F_{k,r}(x) + x^3 \sum_{r \geq 1} (r+2) F_{k,r}(x) + \cdots$$

$$= x \sum_{r\geq 1} r F_{k,r}(x) + x^2 \sum_{r\geq 1} r F_{k,r}(x) + x^2 \sum_{r\geq 1} F_{k,r}(x) + x^3 \sum_{r\geq 1} r F_{k,r}(x) +$$

$$x^3 \sum F_{k,r}(x) + \cdots$$

$$= \sum_{r\geq 1} r F_{k,r}(x) (x + x^2 + x^3 + \cdots) + \sum_{r\geq 1} F_{k,r}(x) (x^2 + x^3 + \cdots)$$

$$= V_k(x) \left(\frac{x}{1-x} \right) + U_k(x) \frac{x^2}{(1-x)^2}$$

$$\Leftrightarrow u_{k+1}(x) = \left(\frac{1}{1-x} u_k(x) + \frac{x^2}{(1-x)^3} u_{k-1}(x)\right) \frac{x}{1-x} + u_k(x) \frac{x^2}{(1-x)^2}$$

$$= \frac{x}{(1-x)^2} u_k(x) + \frac{x^2}{(1-x)^2} u_k(x) + \frac{x^3}{(1-x)^4} u_{k-1}(x)$$

$$= \frac{x(1+x)}{(1-x)^2} u_k(x) + \frac{x^3}{(1-x)^4} u_{k-1}(x)$$

$$\Leftrightarrow \frac{1-x}{x} u_{k+1}(x) = \frac{1+x}{1-x} u_k(x) + \frac{x^2}{(1-x)^3} u_{k-1}(x)$$

$$\Leftrightarrow \frac{1-x}{x} u_{k+1}(x) - \frac{1+x}{1-x} u_k(x) - \frac{x^2}{(1-x)^3} u_{k-1}(x) = 0, \quad \text{----} (7)$$

Where $k \geq 1$ along with the initial data $u_0(x) = 0$, $u_1(x) = \frac{x}{1-x}$

Now to solve (7) we introduce the g.f. $\phi(x,y) = \sum_{k \geq 0} u_k(x) y^k$

Applying the snake-oil method to eqn (7)

$$\sum_{k \geq 1} \frac{1-x}{x} u_{k+1}(x) y^k - \sum_{k \geq 1} \frac{1+x}{1-x} u_k(x) y^k - \sum_{k \geq 1} \frac{x^2}{(1-x)^3} u_{k-1}(x) y^k = 0$$

$$\Leftrightarrow \frac{1-x}{x} \sum_{k \geq 1} u_{k+1}(x) y^k - \frac{1+x}{1-x} \sum_{k \geq 1} u_k(x) y^k - \frac{x^2}{(1-x)^3} \sum_{k \geq 1} u_{k-1}(x) y^k = 0$$

$$\Leftrightarrow \frac{1-x}{xy} \sum_{k \geq 1} u_{k+1}(x) y^{k+1} - \frac{1+x}{1-x} \sum_{k \geq 1} u_k(x) y^k - \frac{x^2 y}{(1-x)^3} \sum_{k \geq 1} u_{k-1}(x) y^{k-1} = 0$$

$$\Leftrightarrow \frac{1-x}{xy} \left[\phi(x,y) - u_1(x) y\right] - \frac{1+x}{1-x} \phi(x,y) - \frac{x^2 y}{(1-x)^3} \phi(x,y) = 0$$

$$\Leftrightarrow \frac{1-x}{xy} \left(\phi(x,y) - \frac{xy}{1-x}\right) - \frac{1+x}{1-x} \phi(x,y) - \frac{x^2 y}{(1-x)^3} \phi(x,y) = 0$$

$$\Leftrightarrow \phi(x,y) \left[\frac{1-x}{xy} - \frac{1+x}{1-x} - \frac{x^2 y}{(1-x)^3}\right] - 1 = 0$$

$$\Leftrightarrow \phi(x,y) \left(\frac{(1-x)^4 - xy(1+x)(1-x)^2 - x^3 y^2}{xy(1-x)^3}\right) = 1$$

$$\Leftrightarrow \phi(x,y) = \frac{xy(1-x)^3}{(1-x)^4 - xy(1+x)(1-x)^2 - x^3 y^2}$$

$$\Leftrightarrow \phi(x,y) = \frac{xy(1-x)^3}{(1-x)^4 - xy(1-x-x^2+x^3+x^2 y)} \quad \text{----} (8)$$

$$\phi(x,y) = \sum_{k \geq 0} u_k(x) y^k = \sum_{n,k,r} f(n,k,r) x^n y^k$$

Note: The sum over r has no variable attached to it; it acts directly on $f(n,k,r)$ and yields the number of HC-polyominoes of n cells and k rows, without regards to how many cells are in the top row.

Thus, we can take $g(n,k) = f(n,k,r)$

$$\Rightarrow \sum_{n,k} g(n,k)\, x^n y^k = \frac{xy(1-x)^3}{(1-x)^4 - xy(1-x-x^2+x^3+x^2y)}$$

perhaps we are interested only in the total number of HC-polyominoes, and we don't need to know the number of rows. In that case, we let $y=1$, in eqn ⑧ and we find the ff result.

Theorem (D. Klarner)

If $f(n)$ is the number of n-celled HC-polyominoes, then

$$\sum_{n\geq 1} f(n)\, x^n = \frac{x(1-x)^3}{1-5x+7x^2-4x^3}$$

$$= x + 2x^2 + 6x^3 + 19x^4 + 61x^5 + 196x^6 + 629x^7 + \cdots$$