Counting Latin square

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Defn: A latin square of order n is a $n \times n$ array filled with n distinct symbols (by convention $\{1, 2, \ldots, n\}$), such that no symbol is repeated twice in any row or column.

- **Example:** The latin squares of order 2 are

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Proposition: Latin squares exist for all n.

Proof: observe that

$$\begin{bmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ n & 1 & \cdots & n-2 & n-1 \end{bmatrix}$$

is a latin square of order n.

Que: How many latin squares exist of a given order n?

The first few terms of the number of Latin squares are

1, 2, 12, 576, 161280, ...

Instead of finding the number of all Latin squares, we study the concept of partial Latin squares.

Defn: A partial latin square of order n is a $n \times n$ array where each cell is filled with either blanks or symbols $\{1, 2, \ldots, n\}$, such that no symbol is repeated twice in any row or column.

- **example:** The following are partial latin squares of order 3.

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}, \quad \begin{bmatrix} & & 3 \\ 2 & 3 & \\ 3 & 1 & 2 \end{bmatrix}$$

Que: When can we complete them into filled-in latin squares?

- For instance, $\begin{bmatrix} & & 3 \\ 2 & 3 & \\ 3 & 1 & 2 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

But, $\begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}$ cannot be completed into a Latin square.

Que: Suppose that we have a $n \times n$ partial Latin Square L in which we have already filled in the first $n-1$ rows. Can we always complete this partial Latin Square to a complete Latin Square i.e. can we always fill in the last row?

Proof: Consider the following simple algorithm for filling in our partial Latin Square:

- Look at row n.
- For each cell (n, i) in row n, look at column i.
- There are $n-1$ distinct symbols in column i, and therefore precisely one symbol s that is not present in column i. Write symbol s in the cell (n, i).

We claim that this algorithm creates a Latin Square. To prove this, it suffices to check whether any symbols are repeated in any row or column. By construction, we know that our choice of symbol s does not cause repetition of symbols in any column; as well as we know that no symbol is repeated in any row other than possibly the n-th row, because we started with a partial Latin Square. Therefore, it suffices to check the n-th row for any repeated symbols.

To do this, proceed by construction; i.e. suppose not, that there are two cells in the bottom row such that we have placed the same symbols in those two cells. This means that there is some symbol s that we have never written in our last row. But this means that this symbol s is used somewhere in all n columns within the first $n-1$ rows, which forces some row in those first $n-1$ to contain two copies of s, a contradiction.

Therefore, our algorithm works!

Defn: A $n \times n$ Latin ~~Square~~ rectangle is a $n \times n$ partial Latin Square in which the first k rows are completely filled and the remaining $n-k$ rows are completely empty, for some value of k.

Theorem: Every Latin rectangle can be completed to a partial Latin Square.

In order to prove this theorem, we use Hall's marriage theorem.

Remark: Take a bipartite graph $G = (V, E)$ with bipartition $V = A \cup B$.
Then the following conditions are equivalent:

- G has a perfect matching: i.e., there is a way to pair off all of the elements of A with the elements of B
- (Hall's condition): For any subset $X \subset A$ or $X \subset B$, if $N(X)$ denotes the neighbors of X, then $|X| \leq |N(X)|$. In other words any collection of vertices from one side of our bipartition cannot have less neighbors on the other side than it has elements.

Thus, by Hall's marriage theorem, there is a perfect matching in G. Deleting it from G leaves a graph in w/c every vertex has degree $k-1$, where $\deg(v) = k$, $\forall v \in V(G)$. Therefore, we can repeat this process until we get the desired k disjoint perfect matchings.

Now, given a $n \times n$ Latin rectangle L with its first k rows filled, we can create a bipartite graph $G = (V, E)$ with $V = A \cup B$, E defined as follows:

- $A = \{1, 2, \cdots, n\}$.
- $B = \{B_1, B_2, \cdots, B_n\}$
- Associate to each B_j the collection of elements that don't occur in column j. Draw an edge from $i \in A$ to $B_j \in B$ iff i is one of those symbols that does not occur in B_j.

By construction, the degree of any B_j is just the number of elts that don't occur in a given column: i.e., $n-k$. As well, the degree of any $i \in A$ is just the number of columns that i doesn't show up, which is also $n-k$; so this is a bipartite graph in which every vertex has degree k.
We claim that this graph satisfies Hall's marriage theorem.
To see why, pick any subset $X \subset A$ or $X \subset B$ of size n. Because every vertex in G has degree $n-k$, there are $n(n-k)$ distinct edges leaving X and entering $N(X)$. Consequently, as each vertex has degree $n-k$, there must be at least n vertices in $N(X)$ to absorb these edges!

Therefore, $|N(X)| \geq |X|$.

If we apply Hall's marriage theorem, then, we have a perfect matching: i.e, a bijection between elts in $\{1, 2, \dots, n\}$ and columns where they do not occur. Use this perfect matching to fill in row $k+1$, by placing in each column the element given by our bijection that does not occur in that column. This gives another Latin rectangle: by repeating this process, we get a Latin square.